Primary ideals and modules

Def: A proper ideal $Q \subsetneq R$ is primary if $xy \in I$ implies $x \in Q$ or $y^n \in Q$ for some $n \in \mathbb{N}$.
A primary ideal Q is P-primary if $\sqrt{Q} = P$.

So P prime $\Rightarrow$ P is primary.

Claim: If Q is primary, then $\sqrt{Q}$ is prime.

Pf: Suppose $ab \in \sqrt{Q}$. Then $(ab)^n \in Q$, so $a^n \in Q$ and thus $a \in \sqrt{Q}$ or $(b^n)^m \in Q$, in which case $b \in Q$. $\square$

The converse doesn't hold:

Ex: Let $Q = (x^2, xy) \subseteq \mathbb{C}[x,y,z]$.

Then $(x^2, xy) \subseteq (x)$ which is prime, so

$\sqrt{Q} \subseteq (x)$. But $x^2 \in Q$ so $(x) \subseteq \sqrt{Q}$. Thus $\sqrt{Q} = (x)$ which is prime.

However, Q is not primary:

$x \notin Q$ and $y^n \notin Q$ for any n.

Ex: In $\mathbb{Z}$, the primary ideals are those of the form (p^n) for p prime.

This generalizes to any PID, as we'll see later. e.g.

$$(x^2 - x^4) = (x^2) \cap (1-x) \cap (1+x) \quad \text{in} \quad k[x]$$

primary

Our goal is to prove the following:

Theorem (Primary decomposition): Let $J \subseteq R$ be an ideal, and R Noetherian. Then $\exists$ primary $Q_1, \ldots, Q_n$ such that

$$J = Q_1 \cap Q_2 \cap \ldots \cap Q_n.$$

Moreover, we can choose the Q_i to have distinct radicals and none contained in another, in which case the decomposition is unique.

However, we can actually prove something more general about modules, by applying theorems from last section. We first generalize the definition to modules:

Def: A submodule $N \subseteq M$ is <u>primary</u> if $\mathrm{Ass}(M/N)$ has just one element P. In this case, we say N is <u>P-primary</u>.

M is <u>coprimary</u> if $0 \subsetneq M$ is primary.

That is: $N \subsetneq M$ primary $\Leftrightarrow$ M/N is coprimary.

We will soon see this is equivalent to our definition in the case of $I \subsetneq R$ an ideal.

For the remainder of this section, assume R is Noetherian and M a finitely-generated R-module.

Lemma: If $P \subsetneq R$ is prime, and $N_1, \ldots, N_t \subsetneq M$ all P-primary in M, then $\bigcap_i N_i$ is P-primary.

Pf: By induction, we can assume $t = 2$.

Then $M/N_1 \cap N_2 \hookrightarrow M/N_1 \oplus M/N_2$, so recalling how associated primes behave in short exact sequences, we have

$$\mathrm{Ass}\left(M/N_1 \cap N_2\right) \subseteq \mathrm{Ass}\left(M/N_1\right) \oplus \mathrm{Ass}\left(M/N_2\right) \subseteq \mathrm{Ass}\left(M/N_1\right) \cup \mathrm{Ass}\left(M/N_2\right) = \{P\}$$

(using the SES $0 \to M/N_1 \to M/N_1 \oplus M/N_2 \longrightarrow M/N_2 \to 0$).

Since $\mathrm{Ass}\left(M/N_1 \cap N_2\right) \neq \emptyset$, $N_1 \cap N_2$ is P-primary. $\square$

The big theorem about associated primes gives us the following characterization of coprimary modules:

Prop: Let $P \subseteq R$ be prime. The following are equivalent:

a.) M is P-coprimary.

b.) P is minimal over $\text{Ann}\, M$ and every elt not in P is a nonzerodivisor on M. (i.e. $u \notin P$ and $um = 0 \Rightarrow m = 0$)

c.) A power of P annihilates M, and every element not in P is a NZD on M.

Note: If $M = R/I$, $P = \sqrt{I}$, then c.) says that a power of P is in I (which is always true for R Noeth), and if $a \notin \sqrt{I}$, then $ab \notin I$ for any $b \notin I$.

This is equivalent to $ab \in I \Rightarrow a \in \sqrt{I}$ or $b \in I$, so the definition of primary for modules agrees with our def'n for ideals.

Pf of prop:

a.) $\Rightarrow$ b.): P is the only associated prime of M, so it must be minimal over $\text{Ann}\, M$, and $P = \{\text{zero divisors on } M\} \cup \{0\}$.

b.) $\Rightarrow$ c.): We just need to show some power of P is in $\text{Ann}\, M$. First we show it is true when we localize:

Let $P' = PR_p$. Then P' is minimal over $(\operatorname{Ann} M)R_p$, which is contained in $\operatorname{Ann} M_p$.

If $\frac{r}{u} \in \operatorname{Ann} M_p$, then for $m \in M$, $\frac{r}{1} m = 0 \Rightarrow rvm = 0$ for some $v \notin P \Rightarrow rv \in \operatorname{Ann} M \Rightarrow r \in (\operatorname{Ann} M) R_p$.

Thus, P' is minimal over $\operatorname{Ann} M_p$.

$\Rightarrow P' = \sqrt{\operatorname{Ann} M_p}$.

$\Rightarrow$ If $P' = (x_1, \dots, x_m)$ then $\exists\ k_i$ s.t. $x_i^{k_i} \in \operatorname{Ann} M_p$.

$\Rightarrow \exists\ n >> 0$ s.t. $(P')^n \subseteq \operatorname{Ann} M_p$.

Now let $r \in P^n$, $m \in M$. Then $\frac{rm}{1} = 0$, so $\exists\ u \notin P$ s.t. $urm = 0$. But u is a NZD on M, so $rm = 0$. Thus, $P^n \subseteq \operatorname{Ann} M$.

<u>c.) $\Rightarrow$ a.)</u>: $P^n \subseteq \operatorname{Ann} M$, so $P \subseteq \sqrt{\operatorname{Ann} M} = \bigcap_{P_i \supset \operatorname{Ann} M} P_i$

By the second hypothesis, $\operatorname{Ann} M \subseteq P$, so P must be minimal among primes containing $\operatorname{Ann} M$, so $P \in \operatorname{Ass} M$.

The elts outside P are NZD on M, so all associated primes are in $P \Rightarrow M$ is P-coprimary. □

<u>Note:</u> Part b) tells us that M is P-coprimary $\Leftrightarrow P$ is

minimal over $\mathrm{Ann}\, M$ and M injects into M_p

So if M is any module and P is minimal over $\mathrm{Ann}M$, then set $M' = \ker(M \to M_p)$.

then $0 \to M' \to M \to M_p$ is exact, so

$$0 \to M'_p \to M_p \xrightarrow{\cong} M_p \text{ is as well.}$$

Thus, M/M' injects into $M_p = M_p/M'_p = (M/M')_p$, so M' is P-primary.

Ex: let $I = (x^2y) \subseteq k[x,y]$, and $M = k[x,y]/I$.

Then the minimal primes over $\mathrm{Ann}M = I$ are (x) and (y).

$$\ker(M \to M_{(x)}) = \{m \in M \mid vm = 0 \text{ for some } v \notin (x)\} = (x^2)$$
$$\ker(M \to M_{(y)}) = (y)$$

Notice: $I = (x^2) \cap (y)$.

However...

Ex: $I = (x^2, xy) \subseteq k[x,y]$, $M = k[x,y]/I$. The only minimal prime over I is (x), and

$$\ker(M \to M_{(x)}) = (x).$$

But $I \neq (x)$.

In the next section, we describe exactly how to find a primary decomposition.